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Some Examples on Equivalence of Tolman and Einstein Energy Momentum Complexes in General Relativity

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Abstract. Through the lens of Tolman and Einstein energy momentum formulations, we conduct a comprehensive analysis of the energy momentum localization in different space times (diagonal and non-diagonal). We obtained that the two formulations (Tolman and Einstein) provide the same results for these space times (diagonal and non-diagonal). However, we further obtained that the super-potentials of Tolman and Einstein are different in general relativity for all diagonal and non-diagonal space times.

Keywords: tolman energy momentum complex, einstein energy momentum complex, general relativity, space time

Introduction

The structure of space time in special relativity is characterized as flat. Within this context, the energy and momentum associated with both matter and non-gravitational field sources can be elegantly encapsulated using a mathematical construct called the energy momentum tensor A_{ν}^{μ}. This tensor provides a concise and comprehensive description of how energy and momentum are distributed and interact within the fabric of space time. Energy momentum complexes are a fascinating area of research in general relativity. That was Einstein, who first of all gave mathematical expression for energy and momentum distribution and tried to show that how energy and momentum are distributed locally. Although Einstein has been formulating the theory of general relativity for over a century, the issue of energy and momentum distribution remains a challenging problem in general relativity that is still unresolved. After Einstein, a number of scientist have tried to resolve the problem of energy and momentum distribution and gave their own definitions. Including definition of Tolman (Ali *et al*., 2022), Weinberg formulation (Radinschi *et al*., 2020), Bergmann Thomson (Aygun and Yilmaz, 2008), Papapetrou (Sharif, 2004) and Landau-lifshitz (Sharif and Nazir, 2008). One major drawback of these formulations is their reliance on specific coordinate systems. Meaningful results are attainable exclusively by employing cartesian coordinate in calculations (Sharif, 2004). Physicists such as Penrose (1982), Moller (1961), Komar (1959), Moller (1958) developed coordinate independent definitions of energy momentum formulations as a response to the limitations imposed by coordinate dependence.

Although the Komar formulation does not rely on Cartesian coordinates, it cannot be applied to non-static space times. The contention made by Moller was that his formulation provides an equivalent total energy momentum values as Einstein's energy momentum formulation for a closed system. Moller's energy momentum expression, however, faced criticism from various sources (Kovacs, 1985; Penrose, 1982; Moller, 1961; Komar, 1959). The conceptual significance of quasi-local masses reported by Penrose (1982). The deficiencies of these quasi-local masses have been the subject of discussion of Virbhadra (1999), Bernstein and Tod (1994) and Bergqvist (1992) as they do not yield consistent output for the kerr metrics and Reissner-Nordstrom. Additionally, the expression proposed by penrose fail to adequately address the kerr metric. Each of these energy momentum prescriptions possesses its own limitations. Thus, these notions of energy momentum prescriptions received substantial criticism in response.

The result of all these prescriptions (coordinate dependent and independent) are sometimes coincide and sometimes disagree for the same space time in general relativity (GR). The credit for revitalizing interest in this approach goes to Virbhadra (1999), Virbhadra (1990a and b). Several investiga-tions have been conducted by different authors to evaluate the

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energy momentum distributions in various space times (Sharif, 2003 and 2002; Yang and Radinschi, 2002; Xulu, 2000a and b; Virbhadra's, 1999) recent paper examined the equivalence of energy momentum prescriptions proposed by Papapetrou, Landau-lifshitz, Weinberg and Einstein for the energy momentum distribution in general non-static spherically symmetric metric. A remarkable discovery was made when, in contrast to previous findings in both asymptotically non-flat and asymptotically flat space times, it was discovered that these formulations show disagreement. Rosen and Virbhadra (1993) as well as Chamarro and Virbhadra (1995) computed the energy momentum of various space times. (Virbhadra, 1999; Virbhadra, 1990a and b) and utilized Einstein's, Weinberg, Papapetrou's and Landau-lifshitz energy momentum formulations to analyze the energy momentum distribution in non-static spherically symmetric space times of the kerr-schild class. His research indicated that all of these approaches yielded comparable energy momentum distributions when compared to the penrose energy- momentum complex. The research conducted by Banerjee and Sen (1997) demonstrated that the overall energy within a Bianchi type-I unverse is uniformly zero throughout. But there are documented cases (Sharif and Fatima 2005a and b; Sharif, 2003; Sharif, 2002) that challenge this assertion.

It has been suggested by certain authors that the Teleparallel theory of gravity (TPT) may provide an alternate approach to solving the issue of energy momentum distribution. Additionally their investigations demonstrate that energy momentum localization is attainable within the framework of TPT. On certain occasion there is agreement between the predictions of both GR and the TPT. By employing the TPT version of Lindau-lifshitz's and Einsteis's energy momentum prescriptions, Vargas (2004) proved that the overall energy of the closed Friedmann-Robertson-Walker universe is zero. The output derived by Vargas concur with the results reported by Rosen (1994), demonstrating consistency between their respective outcomes. The identification of this phenomenon provided an opportunity for numerous authors to examine the energy momentum characteristics of different space times by employing the teleparallel version of various prescriptions. while some space times yielded consistent results across different prescriptions, discrepancies emerged in other cases.

The present study investigates different space times geometries (diagonal and non-diagonal) to analyzing its energy momentum distribution through the Tolman and Einstein expressions in GR. The layout of the paper is as follows: section 2 of the paper outlines the formulations of Tolman and Einstein in GR, section 3 contain the material methods of this research and section 4 contains the findings of the energy momentum distribution in different space times (diagonal and non-diagonal) within the GR, using the two formulations, section 5 provided results and discussion of this research. Finally, section 6 presented a brief conclusion to this research.

Tolman and Einstein energy momentum tensors. This section aims to summarize the Tolman and Einstein formulations utilized in GR for calculating energy momentum tensors.

Tolman energy momentum tensor. In GR, the Tolman definition of energy momentum distribution is represented by the following formula (Ali *et al.*, 2022).

$$\tau_\alpha^\mu = \frac{2}{\kappa}\tau_{\alpha,\sigma}^{\mu\sigma} \quad \text{(1a)}$$

where:

$$\tau^{\mu\nu} = g^{\nu\alpha}\tau_\alpha^\mu = \frac{2}{\kappa}g^{\nu\alpha}\tau_{\alpha,\sigma}^{\mu\sigma} \quad \text{(1b)}$$

$$\tau_\alpha^{\mu\nu} = \frac{\sqrt{-g}}{8}\left(\delta_\alpha^\mu g^{ij}V_{ij}^\nu - 2g^{\mu\kappa}V_{\alpha\kappa}^\nu\right)\text{non-symmetric} \quad \text{(1c)}$$

Einstein energy momentum tensor. In GR, the Einstein definition of energy momentum distribution is represented by the following formula (Ali *et al.*, 2022).

$$\chi_\alpha^\mu = \frac{2}{\kappa}\chi_{\alpha,\sigma}^{[\mu\sigma]} \quad \text{(2a)}$$

$$\chi^{\mu\nu} = g^{\nu\alpha}\chi_\alpha^\mu = \frac{2}{\kappa}g^{\nu\alpha}\chi_{\alpha,\sigma}^{[\mu\sigma]} \quad \text{(2b)}$$

where:

$$\chi_\alpha^{[\mu\sigma]} = \frac{1}{4\sqrt{-g}}g_{\alpha\beta}\lambda_{,\gamma}^{[\mu\nu][\beta\gamma]} \text{ Anti-symmetric} \quad \text{(2c)}$$

and $g = \det(g_{\mu\nu})$

where:
κ is Einstein constant

$$\lambda^{[\mu\nu][\alpha\beta]} = -g(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}) \text{ Anti-symmetric} \quad \text{(3)}$$

$$V_{\nu\alpha}^\mu = \delta_\nu^\mu\Gamma_{i\alpha}^i + \delta_{i\nu}^\mu - 2\Gamma_{\nu\alpha}^\mu \text{ non-symmetric} \quad \text{(4)}$$

where:
$\lambda^{[\mu\nu][\alpha\beta]}$ is represent Landau-lifshitz presccription.
And Γ^{i}_{jk} is used for second kind of christoffel symbol which is equal to the metric

$$\Gamma^{\kappa}_{ij} = \frac{1}{2} g^{\kappa\alpha}(g_{i\alpha,j} + g_{j\alpha,i} - g_{ij,\alpha}) \quad \text{.......................... (5)}$$

Material and Methods

We need to find the energy momentum tensors and super potentials for both Tolman and Einstein prescriptions for any metric in space times by using Mathematica software to show that Einstein and Tolman prescriptions will give same energy momentum tensors and give different super potentials. We will first show generally that Tolman and Einstein's defined energy momentum tensors are the same regardless of the metric. Then we will show that the super potentials defined by Tolman and Einstein are different for any metric. We will make use of Mathematica software to obtain our results.

Spacetime models in focus. We will examine twelve distinct space times and demonstrate that they yield identical results for Tolman and Einstein energy momentum tensors in all instances.

Example 1. We will examine the Bell-Szekeres Metric (Sharif and Nazir, 2008) expressed in cartesian coordinates.

$$ds^2 = \frac{1}{2,} dt^2 - Cos^2\left[A\frac{(t+z)}{2}\theta\frac{(t+z)}{2} + B\frac{(t-z)}{2}\theta\frac{(t-z)}{2}\right] dx^2 - Cos^2[A\frac{(t+z)}{2}\theta\frac{(t+z)}{2} + B\frac{(z-t))}{2}\theta\frac{(t-z)}{2}]dy^2 - \frac{1}{2}dz^2 \quad \text{....... (6)}$$

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$$2\tau^{00}_{3} = \tau^{03}_{0} = \frac{1}{16}(-2BSin[B(t-z)\theta\frac{(t-z)}{2}]\theta\frac{(t-z)}{2}] + 2ASin[A(t+z)\theta\frac{(t+z)}{2}]\theta\frac{(t+z)}{2} - BtSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t-z)}{2} + BzSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t-z)}{2} + AtSin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t+z)}{2}] + AzSin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t+z)}{2})$$

$$\tau^{01}_{1} = -\tau^{02}_{2} = \frac{1}{16}(-2BSin[A(t+z)\theta\frac{(t+z)}{2}]\theta\frac{(t-z)}{2} - 2ASin[B(t-z)\theta\frac{(t-z)}{2}]\theta\frac{(t+z)}{2} - BtSin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t-z)}{2} + BzSin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t-z)}{2} - AtSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t+z)}{2} - AzSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t+z)}{2})$$

$$\tau^{03}_{3} = \frac{1}{16}(2BSin[B(t-z)\theta\frac{(t-z)}{2}]\theta\frac{(t-z)}{2} + 2ASin[A(t+z)\theta\frac{(t+z)}{2}]\theta\frac{(t+z)}{2} + BtSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t-z)}{2} - BzSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t-z)}{2} + AtSin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t+z)}{2} + AzSin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t+z)}{2})$$

$$\chi^{01}_{1} = -\chi^{02}_{2} = \frac{1}{16}(Sin[B(t-z)\theta\frac{(t-z)}{2}] - Sin[A(t+z)\theta\frac{(t+z)}{2}])(2B\theta\frac{(t-z)}{2} - 2A\theta\frac{(t+z)}{2} + Bt\theta'\frac{(t-z)}{2} - Bz\theta'\frac{(t-z)}{2} - At\theta'\frac{(t+z)}{2} - Az\theta'\frac{(t+z)}{2})$$

$$\chi^{03}_{0} = -\chi^{03}_{3} = \frac{1}{8}(-2BSin[B(t-z)\theta\frac{(t-z)}{2}]\theta\frac{(t-z)}{2} + 2ASin[A(t+z)\theta\frac{(t+z)}{2}]\theta\frac{(t+z)}{2} - BtSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t-z)}{2} + BzSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t-z)}{2} + AtSin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t+z)}{2} + AzSin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t+z)}{2}) \quad \text{....... (7)}$$

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$$\chi^{0}_{0} = \tau^{0}_{0} = \frac{1}{8\kappa}(4B^2Cos[B(t-z)\theta\frac{(t-z)}{2}](\theta\frac{(t-z)}{2})^2 + 4A^2Cos[A(t+z)\theta\frac{(t+z)}{2}](\theta\frac{(t+z)}{2})^2 + 4BSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t-z)}{2} + 4B^2(t-z)Cos[B(t-z)\theta\frac{(t-z)}{2}]\theta\frac{(t-z)}{2}\theta'\frac{(t-z)}{2} + B^2t^2Cos[B(t-z)\theta\frac{(t-z)}{2}](\theta'\frac{(t-z)}{2})^2 - 2B^2tzCos[B(t-z)\theta\frac{(t-z)}{2}](\theta'\frac{(t-z)}{2})^2 + B^2z^2Cos[B(t-z)\theta\frac{(t-z)}{2}](\theta'\frac{(t-z)}{2})^2 + 4ASin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t+z)}{2} + 4A^2(t+z)Cos[A(t+z)\theta\frac{(t+z)}{2}]\theta\frac{(t+z)}{2}\theta'\frac{(t+z)}{2} + A^2t^2Cos[A(t+z)\theta\frac{(t+z)}{2}](\theta'\frac{(t+z)}{2})^2 + 2A^2tzCos[A(t+z)\theta\frac{(t+z)}{2}](\theta'\frac{(t+z)}{2})^2 + A^2z^2Cos[A(t+z)\theta\frac{(t+z)}{2}](\theta'\frac{(t+z)}{2})^2 + BtSin[B(t-z)\theta\frac{(t-z)}{2}]\theta''\frac{(t-z)}{2} - BzSin[B(t-$$

$z)\theta\frac{(t-z)}{2}]\theta''\frac{(t-z)}{2} + AtSin[A(t+z)\theta\frac{(t+z)}{2}]\theta''\frac{(t+z)}{2} + AzSin[A(t+z)\theta\frac{(t+z)}{2}]\theta''\frac{(t+z)}{2})$

$\chi_3^0 = \tau_3^0 = \frac{1}{8\kappa}(-4B^2Cos[B(t-z)\theta\frac{(t-z)}{2}](\theta\frac{(t-z)}{2})^2 + 4A^2Cos[A(t+z)\theta\frac{(t+z)}{2}](\theta\frac{(t+z)}{2})^2 - BSin[B(t-z)\theta\frac{(t-z)}{2}]\theta'\frac{(t-z)}{2}] - 4B^2(t-z)Cos[B(t-z)\theta\frac{(t-z)}{2}]\theta\frac{(t-z)}{2}\theta'\frac{(t-z)}{2} - B^2t^2Cos[B(t-z)\theta\frac{(t-z)}{2}](\theta'\frac{(t-z)}{2})^2 + 2B^2tzCos[B(t-z)\theta\frac{(t-z)}{2}](\theta'\frac{(t-z)}{2})^2 - B^2z^2Cos[B(t-z)\theta\frac{(t-z)}{2}](\theta'\frac{(t-z)}{2})^2 + 4ASin[A(t+z)\theta\frac{(t+z)}{2}]\theta'\frac{(t+z)}{2} + 4A^2(t+z)Cos[A(t+z)\theta\frac{(t+z)}{2}]\theta\frac{(t+z)}{2}\theta'\frac{(t+z)}{2} + A^2t^2Cos[A(t+z)\theta\frac{(t+z)}{2}](\theta'\frac{(t+z)}{2})^2 + A^2tzCos[A(t+z)\theta\frac{(t+z)}{2}](\theta'\frac{(t+z)}{2})^2 + A^2z^2Cos[A(t+z)\theta\frac{(t+z)}{2}](\theta'\frac{(t+z)}{2})^2 - BtSin[B(t-z)\theta\frac{(t-z)}{2}]\theta''\frac{(t-z)}{2} + BzSin[B(t-z)\theta\frac{(t-z)}{2}]\theta''\frac{(t-z)}{2} + AtSin[A(t+z)\theta\frac{(t+z)}{2}]\theta''\frac{(t+z)}{2} + AzSin[A(t+z)\theta\frac{(t+z)}{2}]\theta''\frac{(t+z)}{2})$ $\chi_i^0 = \tau_i^0 = 0\ i = 1,2$ .... (8)

Example 2. The special class of ferrari-lbanez degenerate space time (Sharif and Azam, 2007) is given by

$ds^2 = (1+\delta sint)^2(dt^2 - dz^2) - \frac{(1-\delta sint)}{(1+\delta sint)}dx^2 - cos^2z(1+\delta sint)^2dy^2$ .... (9)

where:
δ is an arbitrary constant.

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$\tau_3^{00} = -\frac{\iota\sqrt{-1+\delta \mathrm{sint}}cos'z}{8\sqrt{\mathrm{cos}z}}$, $\tau_1^{01} = \frac{\iota\sqrt{cosz}\sqrt{-1+\delta\mathrm{sin}t}(-7+5\delta\mathrm{sin}t)\delta\mathrm{sin}'t}{8(-1+\delta\mathrm{sin}^2t)}\tau_3^{03} = -\frac{\iota\sqrt{\mathrm{cos}z}\sqrt{-1+\delta\mathrm{sin}t}(-1+3\delta\mathrm{sin}t)\delta\mathrm{sin}'t}{8(-1+\delta\mathrm{sin}^2t)}\chi_1^{01} = -\frac{(3\iota\sqrt{cosz}\sqrt{-1+\delta\mathrm{sin}t}\delta\mathrm{sin}'t)}{4+4\delta\mathrm{sin}t}$, $\chi_2^{02} = 2\tau_2^{02} = -\frac{\iota\sqrt{\mathrm{cos}z}\delta\mathrm{sin}'t}{4\sqrt{-1+\delta\mathrm{sin}t}}\chi_0^{03} = \tau_0^{03} = -\frac{\iota\sqrt{-1+\delta\mathrm{sin}t}(1+\delta\mathrm{sin}t)\mathrm{cos}'z}{4\sqrt{\mathrm{cos}z}}$, $\chi_3^{03} = -\frac{\delta^2\mathrm{sin}t\iota\sqrt{\mathrm{cos}z}\mathrm{sin}'t)}{\sqrt{-1+\delta\mathrm{sin}t}(2+2\delta\mathrm{sin}t]}$ .... (10)

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$\chi_0^0 = \tau_0^0 = \frac{\iota\sqrt{-1+\delta\mathrm{sin}t}(1+\delta\mathrm{sin}t)((cos'z)^2-2\mathrm{cos}z(cos''z))}{4\kappa(\mathrm{cos}z)^{\frac{3}{2}}}\chi_3^0 = \tau_3^0 = -\frac{2\iota\delta\mathrm{sin}t\mathrm{cos}']z\delta\mathrm{sin}'t}{\kappa\sqrt{\mathrm{cos}z}\sqrt{-1+\delta\mathrm{sin}t}(4+4\delta\mathrm{sin}t))}$

$\chi_i^0 = \tau_i^0 = 0\ for\ i = 1,2$ ....(11)

Example 3. The szekers class II space time (Aygun *et al*., 2006) is given by

$ds^2 = -dt^2 + Q^2dx^2 + R^2(dy^2 + h^2dz^2)$ ....(12)

here h=h(y), R=R(t) and Q=(x,y,z,t)

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$\tau_1^{00} = \frac{1}{4}hR^2Q_x\ \tau_2^{00} = \frac{1}{4}R^2(Qh' + hQ_y), \tau_3^{00} = \frac{1}{4}hR^2Q_z\tau_1^{01} = -\frac{1}{4}hR(-2QR' + RQ_t)\quad \tau_2^{02} = \tau_3^{03} = \frac{1}{4}hR^2Q_t\chi_1^{01} = hQRR',\quad \chi_0^{02} = \tau_0^{02} = \frac{1}{2}(Qh' + hQ_y)\chi_2^{02} = \frac{1}{2}hR(QR' + RQ_t),\quad \chi_0^{03} = \tau_0^{03} = \frac{Q_z}{2h}\chi_3^{03} = \frac{1}{2}hR(QR' + RQ_t)$ ....(13)

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$\chi_0^0 = \tau_0^0 = \frac{1}{\kappa}\left(Qh'' + \frac{Q_{zz}}{h} + 2h'Q_y + hQ_{yy}\right)\quad \chi_1^0 = \tau_1^0 = \frac{1}{\kappa}(2hRR'Q_x)\chi_2^0 = \tau_2^0 = \frac{1}{\kappa}\left(Rh'(QR' + RQ_t) + h\left(R'Q_y + RQ_{yt}\right)\right)\chi_3^0 = \tau_3^0 = \frac{1}{\kappa}\left(hR(R'Q_z + RQ_{zt})\right)$ ....(14)

Example 4. The general form of the diagonal space time (Korunur *et al*., 2006) is

$ds^2 = -A^2dt^2 + B^2dx^2 + C^2dy^2 + F^2dz^2$ ....(15)

The functions A, B, C and F depend on the variables t, x, y and z.

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$$\tau_1^{00} = \frac{BCF}{4A}\left(-\frac{A_x}{A} + \frac{B_x}{B} + \frac{C_x}{C} + \frac{F_x}{F}\right) \quad \tau_2^1 = \frac{BCF}{4A}\left(-\frac{A_y}{A} + \frac{B_y}{B} + \frac{C_y}{C} + \frac{F_y}{F}\right) \quad \tau_3^{00} = \frac{BCF}{4A}\left(-\frac{A_z}{A} + \frac{B_z}{B} + \frac{C_z}{C} + \frac{F_z}{F}\right) \quad \tau_1^{02} = \frac{BCF}{4A}\left(\frac{A_t}{A} - \frac{B_t}{B} + \frac{C_t}{C} + \frac{F_t}{F}\right)\tau_2^{02} = \frac{BCF}{4A}\left(\frac{A_t}{A} + \frac{B_t}{B} - \frac{C_t}{C} + \frac{F_t}{F}\right) \quad \tau_3^{03} = \frac{BCF}{4A}\left(\frac{A_t}{A} + \frac{B_t}{B} + \frac{C_t}{C} - \frac{F_t}{F}\right)\chi_0^{01} = \tau_0^{01} = \frac{A(FC_x+CF_x)}{2B} \quad \chi_1^{01} = \frac{B(FC_t+CF_t)}{2A}\chi_0^{02} = \tau_0^{02} = \frac{A(FB_y+BF_y)}{2C} \quad \chi_2^{02} = \frac{C(FB_t+BF_t)}{2A}\chi_0^{03} = \tau_0^{03} = \frac{A(CB_z+BC_z)}{2F} \quad \chi_3^{03} = \frac{F(CB_t+BC_t)}{2A} \quad \ldots\ldots (16)$$

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$$\chi_0^0 = \tau_0^0 = \frac{1}{\kappa}\left(\frac{A_z(CB_z+BC_z)}{F} - \frac{A(CB_z+BC_z)F_z}{F^2} + \frac{A(2B_zC_z+CB_{zz}+BC_{zz}}{F} + \frac{A_y(FB_y+BF_y)}{C} - \frac{AC_y(FB_y+BF_y)}{C^2} + \frac{A(2B_yF_y+FB_{yy}+BF_{yy})}{C} + \frac{A_x(FC_x+CF_x)}{B} - \frac{AB_x(FC_x+CF_x)}{B^2} + \frac{A(2C_xF_x+FC_{xx}+CF_{xx})}{B}\right)\chi_1^0 = \tau_1^0 = \frac{1}{\kappa A^2}(AB_x(FC_t + CF_t) + B(-CA_xF_t + F(-A_xC_t + AC_{tx}) + A(F_xC_t + C_xF_t + CF_{tx})))\chi_2^0 = \tau_2^0 = \frac{1}{\kappa A^2}(AC_y(FB_t + BF_t) + C(-BA_yF_t + F(-A_yB_t + AB_{ty}) + A(F_yB_t + B_yF_t + BF_{ty})))\chi_3^0 = \tau_3^0 = \frac{1}{\kappa A^2}(-FA_z(CB_t + BC_t) + AF_z(CB_t + BC_t) + AF(C_zB_t + B_zC_t + CB_{tz} + BC_{tz})) \quad \ldots\ldots (17)$$

Example 5. Here we consider the general Bianchi type diagonal space time (Aygun *et al.*, 2006)

$$ds^2 = -dt^2 + A^2dx^2 + B^2e^{2\alpha x}dy^2 + F^2e^{2\beta x}dz^2) \quad \ldots\ldots (18)$$

The functions A, B and F depend on the variable t only and α, β are constants.

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$$\tau_1^{01} = \frac{1}{4}e^{\frac{\alpha x+\beta x}{2}}\left(AFB' + B(-FA' + AF')\right)$$

$$\tau_3^{03} = \frac{1}{4}e^{\frac{\alpha x+\beta x}{2}}\left(AFB' + B(FA' - AF')\right)$$

$$\chi_1^{01} = \frac{1}{2}e^{\frac{\alpha x+\beta x}{2}}A(FB' + BF'),$$

$$\chi_2^{02} = \frac{1}{2}e^{\frac{\alpha x+\beta x}{2}}B(FA' + AF')$$

$$\chi_3^{03} = \frac{1}{2}e^{\frac{\alpha x+\beta x}{2}}F(BA' + AB') \quad \ldots\ldots (19)$$

By substituting these values into equation (1a) and (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$$\chi_i^0 = \tau_i^0 = 0 \text{fori=0,1,2,3} \quad \ldots\ldots (20)$$

Example 6. By putting $\alpha = \beta = 0$ in eq(18) then the diagonal space time describe the well-known Bianchi type I (Aygun *et al.*, 2006)

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + F^2dz^2 \quad \ldots\ldots (21)$$

The functions A, B and F depend on the variable t only.

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$$\tau_1^{01} = \frac{1}{4}\left(AFB' + B(-FA' + AF')\right),$$

$$\tau_2^{02} = \frac{1}{4}\left(-AFB' + B(FA' + AF')\right)$$

$$\tau_0^{03} = \frac{1}{4}\left(AFB + B(FA' - AF')\right),$$

$$\chi_1^{01} = \frac{1}{2}A(FB' + BF')$$

$$\chi_2^{02} = \frac{1}{2}B(FA' + AF')$$

$$\chi_3^{03} = \frac{1}{2}F(BA' + AB') \quad \ldots\ldots (22)$$

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$$\chi_i^0 = \tau_i^0 = 0 \text{ for i=0,1,2,3} \quad \ldots\ldots (23)$$

Example 7. By putting $\alpha = -1, \beta = 0$ in equation (18) then the diagonal space time describe the well-known Bianchi type III (Aygun *et al.*, 2006)

$$ds^2 = -dt^2 + A^2dx^2 + B^2e^{2x}dy^2 + F^2dz^2 \quad \ldots\ldots (24)$$

The functions A, B and F depend on the variable t only and α is a constant.

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$\tau_1^{00} = -\frac{1}{8}e^{\frac{-x}{2}}ABF, \quad \tau_1^{01} = \frac{1}{4}e^{\frac{-x}{2}}(AFB' + B(-FA' + AF'))\tau_2^{02} = \frac{1}{4}e^{\frac{-x}{2}}(-AFB' + B(FA' + AF')), \quad \tau_3^{03} = \frac{1}{4}e^{\frac{-x}{2}}(AFB' + B(FA' - AF'(t)))\chi_0^{01} = \tau_0^{01} = -\frac{e^{\frac{-x}{2}}BF}{4A}, \quad \chi_1^{01} = \frac{1}{2}e^{\frac{-x}{2}}A(FB' + BF')$
$\chi_2^{02} = \frac{1}{2}e^{\frac{-x}{2}}B(FA' + AF'), \quad \chi_3^{03} = \frac{1}{2}e^{\frac{-x}{2}}F(BA' + AB')$ .... (25)

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$\chi_0^0 = \tau_0^0 = \frac{e^{\frac{-x}{2}}BF}{4\kappa A}, \quad \chi_1^0 = \tau_1^0 = -\frac{e^{\frac{-x}{2}}A(FB'+BF')}{2\kappa}$
$\chi_i^0 = \tau_i^0 = 0$, for i=2,3 .... (26)

Example 8. By putting $\alpha = \beta = -1$ in equation (18) then the diagonal space time describe the well-known Bianchi type V (Aygun *et al.*, 2006).

$ds^2 = -dt^2 + A^2dx^2 + B^2e^{-2x}dy^2 + F^2e^{-2x}dz^2$ .... (27)

The functions A, B and F depend on the variable t only.

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$\tau_1^{00} = -\frac{1}{4}e^{-x}ABF, \quad \tau_1^{01} = \frac{1}{4}e^{-x}(AFB' + B(-FA' + AF'))\tau_2^{02} = \frac{1}{4}e^{-x}(-AFB' + B(FA' + AF')), \quad \tau_3^{03} = \frac{1}{4}e^{-x}(AFB' + B(FA' - AF'))\chi_1^{01} = \frac{1}{2}e^{-x}A(FB' + BF'), \quad \chi_2^{02} = \frac{1}{2}e^{-x}B(FA' + AF')\chi_3^{03} = \frac{1}{2}e^{-x}F(BA' + AB')$ .... (28)

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$\chi_0^0 = \tau_0^0 = \frac{e^{-x}BF}{\kappa A}, \quad \chi_1^0 = \tau_1^0 = -\frac{e^{-x}A(FB'+BF')}{\kappa}\chi_i^0 = \tau_i^0 = 0$, for i=2,3.. (29)

Example 9. By putting $\alpha = -\beta = -1$ in equation (18) then the diagonal space time describe the well-known Bianchi type VI_0 (Aygun *et al.*, 2006).

$ds^2 = -dt^2 + A^2dx^2 + B^2e^{-2x}dy^2 + F^2e^{2x}dz^2$ .... (30)

The functions A, B and F depend on the variable t only.

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$\tau_1^{01} = \frac{1}{4}(AFB' + B(-FA' + AF'), \tau_3^{02} = \frac{1}{4}(-AFB' + B(FA' + AF'))\tau_3^{03} = \frac{1}{4}(AFB' + B(FA' - AF')), \chi_1^{01} = \frac{1}{2}A(FB' + BF')\chi_2^{02} = \frac{1}{2}B(FA' + AF'), \chi_3^{03} = \frac{1}{2}F(BA' + AB')$ .... (31)

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$\chi_i^0 = \tau_i^0 = 0$, for $i = 0,1,2,3$ .... (32)

Example 10. Here we consider the non static plane symmetric diagonal space time (Amir *et al.*, 2012).

$ds^2 = e^{2\nu(t,x)}dt^2 - e^{2\mu(t,x)}dx^2 - e^{\lambda(t,x)}(dy^2 + dz^2)$ .... (33)

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$\chi_0^{01} = \tau_0^{01} = -\frac{1}{2}e^{\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu\lambda_x}\,\tau_1^{00} = -\frac{1}{8}e^{\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu}(2\lambda_x + \mu_x - \nu_x)\tau_1^{01} = -\frac{1}{8}e^{\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu}(2\lambda_t - \mu_t + \nu_t)\,\tau_2^{02} = \tau_3^{03} = -\frac{1}{8}e^{\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu}(\mu_t + \nu_t)\chi_1^{01} = -\frac{1}{2}e^{\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu}\lambda_t\,\chi_2^{02} = \chi_3^{03} = -\frac{1}{4}e^{\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu}(\lambda_t + \mu_t)$ .... (34)

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$\chi_0^0=\tau_0^0=-\frac{e^{\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu}(2\lambda_x^2)+\lambda_x(-\mu_x+\nu_x)+2\lambda_{xx})}{2\kappa}\chi_1^0=$

$\tau_1^0=-\frac{e^{\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu}(2\lambda_x\lambda_t+\mu_x\lambda_t-\nu_x\lambda_t+2\lambda_{tx})}{2\kappa}\chi_i^0=\tau_i^0=$

0 for $i=2{,}3$ ....(35)

Example 11. The cartesian form of weyl metric (Sharif and Fatima, 2005) is given by

$ds^2=e^{2\Psi}dt^2-\frac{e^{2(\gamma-\Psi)}}{\rho^2}(xdx+ydy)^2-$

$\frac{e^{-2\psi}}{\rho^2}(xdy-ydx)^2-e^{2(\gamma-\Psi)}dz^2$........................(36)

The functions Ψ and γ depend on the variables ρ and z

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials componen

$\chi_0^{01}=\tau_0^{01}=$

$-\frac{e^{\frac{1}{2}\gamma}x}{2\sqrt{-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+y^4)}}\chi_0^{02}=$

$\tau_0^{02}=$

$-\frac{e^{\frac{1}{2}\gamma}y}{2\sqrt{-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+y^4)}}\chi_0^{03}=$

$\tau_0^{03}=$

$-\frac{1}{4\sqrt{-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+y^4)}\rho^2}$

$\Big(-e^{\frac{-1}{2}\gamma}\,\big(e^{\gamma}(x^4-2(-1+e^{\gamma})xy^2+2e^{\gamma}x^2y^2+$

$y^4)\gamma_z-2\big(-xy^2+\quad x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+$

$e^{\gamma}(x^4+2xy^2+y^4)\big)\psi_z\big)\Big)\tau_1^{00}=$

$-\frac{e^{\frac{1}{2}\gamma-4\psi}x(2e^{\gamma}x^2+y^2+e^{2\gamma}y^2)}{4\sqrt{-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+y^4)}\rho^2}\tau_2^{00}=$

$-\frac{e^{\frac{1}{2}\gamma-4\psi}y\big((1+e^{2\gamma})x^2+2e^{\gamma}y^2\big)}{4\sqrt{-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+y^4)}\rho^2}\tau_3^{00}=$

$-\frac{1}{8\sqrt{-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+y^4)}\rho^2}$

$(e^{\frac{1}{2}\gamma-4\psi}\,\big(\big(-xy^2+x^2y^2-3e^{2\gamma}(xy^2-x^2y^2)+$

$2e^{\gamma}(x^4+2xy^2+y^4)\big)\gamma_z-4\big(-xy^2+x^2y^2+$

$e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+y^4)\big)\psi_z\big)$ (37)

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$\chi_0^0=\tau_0^0=\frac{1}{4(-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+y^4))^{\frac{3}{2}}\kappa}$

$(e^{\frac{-1}{2}\gamma}(4e^{\gamma}y^2((1+e^{2\gamma})x^2+2e^{\gamma}y^2)+$

$4e^{\gamma}x^2(2e^{\gamma}x^2+y^2+e^{2\gamma}y^2)-8e^{\gamma}(-xy^2+$

$x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+$

$y^4))+\frac{1}{\rho^2}e^{\gamma}(x^4-2(-1+e^{\gamma})xy^2+2e^{\gamma}x^2y^2+$

$y^4)\gamma_z(e^{\gamma}(x^4-2(-1+e^{\gamma})xy^2+2e^{\gamma}x^2y^2+$

$y^4)\gamma_z-2(-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+$

$e^{\gamma}(x^4+2xy^2+y^4))\psi_z+\frac{1}{\rho^2}(-xy^2+x^2y^2+$

$e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+$

$y^4))\gamma_z(e^{\gamma}(x^4-2(-1+e^{\gamma})xy^2+2e^{\gamma}x^2y^2+$

$y^4)\gamma_z-2(-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+$

$e^{\gamma}(x^4+2xy^2+y^4))\psi_z-\frac{1}{\rho^2}2(-xy^2+x^2y^2+$

$e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+2xy^2+$

$y^4))(e^{\gamma}(x^4+(2-4e^{\gamma})xy^2+4e^{\gamma}x^2y^2+y^4)\gamma_z^2-$

$2e^{\gamma}(x^4-2(-1+e^{\gamma})xy^2+2e^{\gamma}x^2y^2+y^4)\gamma_z\psi_z+$

$e^{\gamma}(x^4-2(-1+e^{\gamma})xy^2+2e^{\gamma}x^2y^2+y^4)\gamma_{zz}-$

$2(-xy^2+x^2y^2+e^{2\gamma}(-xy^2+x^2y^2)+e^{\gamma}(x^4+$

$2xy^2+y^4))\psi_{zz}))\chi_i^0=\tau_i^0=0,\ for\ i=1,2,3$ ...(38)

Example 12. we will consider Homogeneous Godel-type Metric (Sharif, 2004) in Cartesian coordinates.

$ds^2=dt^2-\frac{1}{r^2}(xdx+ydy)^2+\frac{1}{r^4}(H^2-D^2)$

$(xdy-ydx)^2-dz^2+\frac{2}{r^2}Hdt(xdy-ydx)$ ....(39)

where H and D are the functions of r only.

Plugging the metric coefficients in to equation (1c) and equation (2c) yields the following Tolman and Einstein super-potentials components.

$$\tau_0^{00} = \frac{(x^4-y^4)H(r)\left(r^2-D^2(r)+H^2(r)\right)\left(-xyr^2+xyD^2(r)\right)}{4((-xy^2+x^2y^2)D^4(r)+D^2(r)(x^4+2xy^2+y^4)r^2-(r^2+H^2(r))(xy^2-x^2y^2)r^2)^{\frac{3}{2}}}$$

$$\tau_1^{00} = [-((\sqrt{(-xy^2+x^2y^2)D^4+D^2(x^4+2xy^2+y^4)r^2 \pm (r^2+H^2)(xy^2-x^2y^2)r^2}\,((-x^2y^4+$$
$$x^3y^4)D^8+D^6(x(-2x^3y^2+3x^4y^2+2xy^4+y^6)r^2-y(xy^3-6x^2y^3+5x^3y^3)H^2)+D^4(2x(x^6-xy^4+$$
$$2x^2(xy^2+y^4))r^4+(8x^4y^2-10x^5y^2+3xy^4-10x^2y^4+3x^3y^4-2xy^6)r^2H^2+3xy^2(xy^2-$$
$$x^2y^2)H^4)+r^2+H^2)(xy^2(-xy^2+x^2y^2)r^6+(xy^4-3x^5y^2+3x^3y^4+x^3(4xy^2-y^4)+x(-xy^4+$$
$$y^6))r^4H^2+(-7x^5y^2+3x^5y^2+xy^2(2xy^2+y^4)+x^3(4xy^2+y^4)-x(3xy^4+y^6))r^2H^4)-$$
$$D^2(-x(-2x^3y^2+3x^4y^2+2xy^4+y^6)r^6+(2x^7-7x^5y^2-6x^2y^4+3x^3y^4+4x^3(3xy^2+y^4)+$$
$$xy^2(3xy^2+y^4))r^4H^2+(10x^4y^2-14x^5y^2+6xy^4+3x^5y^2-14x^2y^4+6x^3y^4+2xy^6-3xy^6)r^2H^4+$$
$$\frac{y(3xy^3-10x^2y^3+7x^3y^3)H^6)))}{(4r^4((xy^2-x^2y^2)D^4-D^2((x^4+2xy^2+y^4)r^2)+r^2+H^2)((xy^2-x^2y^2)r^2)^2)}]$$

$$\tau_2^{00} = [-((\sqrt{(-xy^2+x^2y^2)D^4+D^2\left((x^4+2xy^2+y^4)r^2 \pm (r^2+H^2)\left((xy^2-x^2y^2)r^2\right)\right)}\,((-x^3y^3+$$
$$x^4y^3)D^8+D^6(y(x^6-2xy^4+x^2(2xy^2+3y^4))r^2-x(xy^3-6x^2y^3+5x^3y^3)H^2)+D^4(2y(-x^3y^2+$$
$$2x^4y^2+2xy^4+y^6)r^4+(x^6y+3x^2y^3-3x^6y-10x^3y^3+3x^4y^3+8xy^5-10x^2y^5)r^2H^2+$$
$$3x^2y(xy^2-x^2y^2)H^4)+(r^2+H^2)(x^2y(-xy^2+x^2y^2)r^6+(x^2y^3+x^6y+2x^4y^3+4xy^5-x^2y(xy^2+$$
$$3y^4))r^4H^2+(x^4y^3+4xy^5+x(2xy^3-7xy^5)+x^2(-3xy^3+3y^5))r^2H^4+D^2(y(x^6-2xy^4+$$
$$x^2(2xy^2+3y^4))r^6-(x^6y+3x^2y^3-6x^3y^3+7x^4y^3+12xy^5-7x^2y^5+2y^7)r^4H^2+(x^6y-6x^4y^3-$$
$$10xy^5+x(-6xy^3+\frac{14xy^5)+x^2(14xy^3-3y^5))r^2H^4+x(-3xy^3+10x^2y^3-7x^3y^3)H^6)))}{(4r^4((xy^2-x^2y^2)D^4-D^2((x^4+2xy^2+y^4)r^2+(r^2+H^2)((xy^2-x^2y^2)r^2+)^2))}]\tau_0^{01} = \frac{x}{y}\tau_0^{02} =$$
$$\frac{x\left(-2D^2+H^2\right)}{4\sqrt{(-xy^2+x^2y^2)D^4+D^2((x^4+2xy^2+y^4)r^2-(r^2+H^2)((xy^2-x^2y^2)r^2)}}$$

$$\tau_1^{01} = [-((H\sqrt{(-xy^2+x^2y^2)D^4+D^2((x^4+2xy^2+y^4)r^2-(r^2+H^2)((xy^2-x^2y^2)r^2)}((-2xy^3+$$
$$4x^2y^3-2x^3y^3)D^6+D^4(-(-x^5y-6xy^3+8x^2y^3+x^3y^3+2xy^5)r^2+3(2xy^3-5x^2y^3+3x^3y^3)H^2)+$$
$$(r^2+H^2)((2xy^3+x^5y- \quad x^3y^3)r^4+(4xy^3-7x^2y^3+3x^3y^3)r^2H^2)+D^2((-2x^5y-6xy^3+$$
$$4x^2y^3)r^4\,\frac{(-12xy^3+x^5y+22x^2y^3-7x^3y^3+2xy^5)r^2H^2-3(2xy^3-6x^2y^3+4x^3y^3)H^4)))}{\left(4r^2\left((xy^2-x^2y^2)D^4-D^2\left((x^4+2xy^2+y^4)r^2+r^2+H^2\right)\left((xy^2-x^2y^2)r^2\right)\right)^2\right)}]$$

$$\tau_2^{01} = [-((H\sqrt{(-xy^2+x^2y^2)D^4+D^2((x^4+2xy^2+y^4)r^2-(r^2+H^2)((xy^2-x^2y^2)r^2)}(D^4(y(x^4y+$$
$$4xy^3-3x^2y^3)r^2+(x^3y^2-x^4y^2)H^2)+(r^2+H^2)(y(x^4y+4xy^3-3x^2y^3)r^4+(-x^4y^2+4xy^4-$$
$$7x^2y^4+x^2(xy^2+3y^4))r^2H^2)-D^2(y(x^4y+3x^4y+8xy^3-2x^2y^3+2y^5)r^4+(-x^4y^2+8xy^4-$$
$$9x^2y^4\,\frac{2x^2(xy^2+y^4))r^2H^2+x^2(2xy^2-2x^2y^2)H^4)))}{(4r^2((xy^2-x^2y^2)D^4-D^2((x^4+2xy^2+y^4)r^2+(r^2+H^2)((xy^2-x^2y^2)r^2))^2))}]$$

$$\tau_1^{02} = [(H\sqrt{(-xy^2+x^2y^2)D^4+D^2\left((x^4+2xy^2+y^4)r^2\right)-(r^2+H^2)\left((xy^2-x^2y^2)r^2\right)}$$

$$(D^4(x(4x^2y^2-2x^3y^2-x^3y^2+xy^4)r^2+y^2(xy^2-x^2y^2)H^2)-(r^2+ \quad H^2)(x(-4x^2y^2+3x^3y^2-$$
$$xy^4)r^4+(7x^4y^2-3x^4y^2-xy^4+x^2(-4xy^2+ \quad y^4))r^2H^2)-D^2(x(2x^5+8x^2y^2-2x^3y^2+xy^4+$$
$$3xy^4)r^4+(8x^3y^2-\frac{9x^4y^2+2x^4y^2+2xy^4-x^2y^4)r^2H^2+y^2(2xy^2-2x^2y^2)H^4)))}{(4r^2((xy^2-x^2y^2)D^4-D^2((x^4+2xy^2+y^4)r^2)+(r^2+H^2)((xy^2-x^2y^2)r^2))^2)}]$$

$\tau_2^{02} = [(H\sqrt{(-xy^2 + x^2y^2)D^4 + D^2((x^4 + 2xy^2 + y^4)r^2) - (r^2 + H^2)((xy^2 - x^2y^2)r^2)}\ ((-2xy^3 + 4x^2y^3 + x^3y^3 - 3x^3y^3)D^6 + D^4((6xy^3 - 2x^5y - 8x^2y^3 - x^3y^3 + 2xy^5 - 3xy^5)r^2 + 3(2xy^3 - 5x^2y^3 + 3x^3y^3)H^2) + (r^2 + H^2)((2xy^3 - x^3y^3 + xy^5)r^4 + (4xy^3 - 7x^2y^3 + 3x^3y^3)r^2H^2) + D^2((-6xy^3 + 4x^2y^3 - xy^5)r^4 + (-12xy^3 + 2x^5y + 22x^2y^3 - 7x^3y^3 - 3xy^5 + 4xy^5)r^2H^2 - 3(2xy^3 - \frac{6x^2y^3+4x^3y^3)H^4)))}{(4r^2((xy^2-x^2y^2)D^4-D^2((x^4+2xy^2+y^4)r^2)+(r^2+H^2)((xy^2-x^2y^2)r^2))^2)}]\tau_3^{03} = -\frac{(x^4-y^4)H(r^2-D^2+H^2)(-xyr^2+xyD^2)}{4((-xy^2+x^2y^2)D^4+D^2((x^4+2xy^2+y^4)r^2)-(r^2+H^2)((xy^2-x^2y^2)r^2))^{\frac{3}{2}}}\chi_0^{01} = \frac{x(-2D^2+H^2)}{4\sqrt{(-xy^2+x^2y^2)D^4+D^2(x^4+2xy^2+y^4)r^2-(r^2+H^2)(xy^2-x^2y^2)r^2}}\chi_1^{01} = -\chi_2^{02} = \frac{H(r)(xyr^2+xyD^2-xyH^2)}{4r^2\sqrt{(-xy^2+x^2y^2)D^4+D^2(x^4+2xy^2+y^4)r^2-(r^2+H^2)(xy^2-x^2y^2)r^2}}\chi_2^{01} = \frac{H(y^2r^2-x^2D^2+x^2H^2)}{4r^2\sqrt{(-xy^2+x^2y^2)D^4+D^2(x^4+2xy^2+y^4)r^2-(r^2+H^2)(xy^2-x^2y^2)r^2}}\chi_0^{02} = \frac{y(-2D^2+H^2)}{4\sqrt{(-xy^2+x^2y^2)D^4+D^2(x^4+2xy^2+y^4)r^2-(r^2+H^2)(xy^2-x^2y^2)r^2}}\chi_1^{02} = -\frac{H(x^2r^2-y^2D^2+y^2H^2)}{4r^2\sqrt{(-xy^2+x^2y^2)D^4+D^2(x^4+2xy^2+y^4)r^2-(r^2+H^2)(xy^2-x^2y^2)r^2}}$ .... (40)

By substituting these values into equation (1a) and equation (2a), we derived the corresponding components of the Tolman and Einstein energy momentum tensors.

$\chi_0^0 = \tau_0^0 = -\left[\frac{((xy(2D^2-H^2)(r^2-D^2+H^2)(xyr^2-xyD^2))}{(\kappa\sqrt{(-xy^2+x^2y^2)D^4+D^2((x^4+2xy^2+y^4)r^2)-(r^2+H^2)((xy^2-x^2y^2)r^2)}}\right] \left[\frac{1}{((xy^2-x^2y^2)D^4-D^2((x^4+2xy^2+y^4)r^2)+(r^2+H^2)((xy^2-x^2y^2)r^2)))}\right]\chi_1^0 =$
$\tau_1^0 = [(H(y(-4xy^2 + 2x^2y^2)D^6 + D^4((2x^4y + x^4y + 8xy^3 - x^2y^3 + 2y^5)r^2 - y(-11xy^2 + 7x^2y^2)H^2) - D^2((2x^4y + 4xy^3 - 4x^2y^3)r^4 + (3x^4y + 14xy^3 - 5x^2y^3 + 2y^5)r^2H^2 + y(10xy^2 - 8x^2y^2)H^4) + \frac{(r^2+H^2)(xy(x^3-xy^2)r^4+(3xy^3-x^2y^3)r^2H^2+y(3xy^2-3x^2y^2)r^4)))}{(2r^2\kappa((-xy^2+x^2y^2)D^4+D^2((x^4+2xy^2+y^4)r^2)-(r^2+H^2)((xy^2-x^2y^2)r^2))^{\frac{3}{2}})}]\chi_2^0 =$
$\tau_2^0 = [(H(x(-4xy^2 + 2x^2y^2)D^6 + D^4((2x^5 + 8x^2y^2 - x^3y^2 + 3xy^4)r^2 - x(-11xy^2 + 7x^2y^2)H^2) - D^2((4x^2y^2 - 4x^3y^2 + 2xy^4)r^4 + (2x^5 + 14x^2y^2 - 5x^3y^2 + 3xy^4)r^2H^2 + x(10xy^2 - 8x^2y^2)H^4) - \frac{(r^2+H^2)(xy(x^2y-y^3)r^4+(-3x^2y^2+x^3y^2)r^2H^2-x(3xy^2-3x^2y^2)H^4)))}{(2r^2\kappa\sqrt{(-xy^2+x^2y^2)D^4+D^2((x^4+2xy^2+y^4)r^2)-(r^2+H^2)((xy^2-x^2y^2)r^2)}}] [\frac{1}{((xy^2-x^2y^2)D^4-D^2((x^4+2xy^2+y^4)r^2)+(r^2+H^2)((xy^2-x^2y^2)r^2)))}]$
$\chi_3^0 = \tau_3^0 = 0,$ .... (41)

Results and Discussion

The energy with Einstein's pseudo energy momentum tensor agrees with a total energy of massive particles with gravitational interactions through the Newtonian potential and this is conserved (Aoki *et al.*, 2023). They finally discuss an implication from a fact that there exist two conserved quantities, energy and gravitational charge, in general relativity. Cosmological observations increased precision mandates the extension of the general theory of relativity (Giacomini *et al.*, 2017). Like Galileon gravity which has drawn the attention of the scientific society in the last few years the modified theories of gravity also get more attention of scientist (Dimakis *et al.*, 2017). The issue of energy momentum distribution in general relativity has been a source of controversy and has remained under investigation since the inception of the theory. The issue has captured the keen interest of scientists who have dedicated substantial efforts towards its resolution. Aygün and Aktaş (2023) checked the Einstein, Bergmann–Thomson and Landau–Lifshitz energy momentum distributions for Texture metric in Teleparallel gravitation (TG) theory. As we know that these energy momentum complexes have different definitions, Einstein and Bergmann–Thomson energy distributions give identical output for the Texture metric in TG, while the output for Landau–Lifshitz is different. For all types of space times (diagonal and non-diagonal) in the context of general relativity, it has been proven that the Tolman and Einstein energy momentum tensors are identical, as presented in this research. This research showed that Tolman and Einstein super-potentials are different in GR for diagonal and non-diagonal space times. Section 2, presents a brief introduction to the Tolman and Einstein energy momentum expression within the framework of GR. In section 3, a set of space times is chosen as a basis for establishing our findings. In example 1 we consider Bell szeker metric to calculate the energy momentum tensors for Tolman and Einstein prescriptions and we see that the results of both the prescriptions are same for this metric. In example 2 the special class of ferrari-lbanez degenerate space time is consider and the results for energy momentum tensors are same for both the prescriptions. The szekers class II space time is consider in the example 3 and the results of energy momentum tensors are same for both Tolman and Einstein complexes. In example 4 we take general form of the diagonal space time in which the function A, B, C, F are function of t, x, y and z and the results of energy momentum tensors are same for both energy momentum distributions. In example 5 we consider the general Bianchi type diagonal space time and see that the results of Tolman and Einstein energy momentum tensors are same. Fram example 6 to example 9 we consider different values of α and β of space time to get bianchi type I bianchi type III bianchi type V and bianchi type VI_0 space times and see that the results of Tolman and Einstein energy momentum tensors are same for every space time. We consider non static plane symmetric diagonal space time in example 10 to verify that energy momentum tensors of Tolman and Einstein give same results. In example 11 we take weyl metric and see that the results are equivalent for both the prescriptions. Finally Homogeneous Godel-type Metric is taken in example 12 and the results for energy momentum tensors are same for this metric. The physical relevance of the equality between Tolman and Einstein energy momentum tensors lies in the consistent description of energy and momentum within gravitational fields. These tensors are indeed equal, it suggests that different theoretical approaches or calculations result in the same physical interpretation of energy and momentum in the context of general relativity.

Conclusion

The overall conclusion to this research is that in general relativity the complexes of Tolman and Einstein give same values for energy momentum distribution for any metric (diagonal and non-diagonal). We additionally note that the super-potentials for all the different space times (diagonal and non-diagonal) the Tolman and Einstein prescriptions give different results in general in general relativity.

Equating of Tolman and Einstein's energy momentum tensors provide a deeper insight into the physical interpretation of energy and momentum within gravitational fields. Understanding how these different formulations converge helps in grasping the true nature of energy momentum distribution in curved space time. Overall, the confirmation that Tolman and Einstein's energy momentum tensors give identical results across diverse space time geometries would have far-reaching implications for our understanding of gravity and the structure of the universe. It would not only advance theoretical physics but also have potential applications in various scientific and technological domains.

Conflict of Interest. The authors declare that they have no conflict of interest.

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