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Fractional Differential Equations and Chebyshev Polynomials and Their Numerical Solution

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Abstract. In this paper we present an advanced set of polynomials that can be generalized to Chebyshev polynomials. Some basic properties of Chebyshev polynomials and their variables, as well as formulas related to generalized polynomials, will be presented. Orthogonal polynomials are used to solve linear polynomial fractional differential equations (FDEs) which arise from many applications. The proposed algorithm was deduced using a novel method of the power formula for Chebyshev polynomials and the Galerkin formula. The method transforms FDE differential equations with initial or boundary conditions into a system of linear equations that can be solved efficiently and accurately by solving the appropriate numerical solution. The paper includes some examples and comparisons with other methods to prove the effectiveness and usefulness of the proposed algorithm.

Keywords: approximation techniques, numerical algorithms, polynomial fractional, orthogonal polynomials, chebyshev polynomial

Introduction

Sometimes there are no analytical solutions to differential equations. The goal is to find solutions to possible equations and this is what motivated us to design the numerical approximation on irregular grid points for some fractional derivatives, (Maurya and Singh, 2023). A stable and highly adaptive implicit scheme for the time partial propagation wave equations is developed using the order discretization of the Caputo derivative in the time domain. There may be a need for approximate methods based on algorithms in some programs and programming languages. For examples of different numerical methods for solving FDEs, (Li and Zhao, 2010). They are used differently to model many models that we need in many fields, such as mechanical models (Hosseini *et al.*, 2021). Studies have focused on dealing with FDEs due to their importance in several areas. Spectroscopic methods were relied upon to solve such problems (Sun *et al.*, 2019). We take a different approach to treating certain types of FDEs, numerical algorithms were used in (Srivastava and Rai, 2010), to address the partial polynomial propagation wave equation. In (Dehghan *et al.*, 2015) we follow different approaches to treat the polynomial fractional wave equation. Numerical methods used in differential equations have been classified into different types. In contrast to the spectroscopic method, which is more widespread, finite difference and elements have more limited proofs. Problems involving difficult geometries and waveforms are suitable for finite element methods. Spectroscopic methods provide resolution in the form of multi-step finite difference diagrams to solve one-dimensional (1D) and two-dimensional (2D) partial differential models of electromagnetic waves arising from dielectric media containing both Initial boundary conditions and Dirichlet boundaries. Fractional Caputo derivatives are estimated in time by a difference-of-order scheme, fractional differential equations (FDEs) describe electro-magnetic fields and waves (EFWs) in broad fields of dielectric media. Recently, many researches have shown interest in analyzing the existence and applications of FDEs in electromagnetic theory (Maurya, 2021; Maurya *et al.*, 2020). We find several types of spectral methods for solving integral and differential equations. For the Tau and Galerkin spectral methods, we choose two types of basic functions, respectively with empirical functions. When we use the Galerkin method, we choose empirical functions such that all empirical functions that verify the basic conditions are the same as the outcome functions for

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example, (Alsuyuti *et al*., 2019; Ezz-Eldien *et al*., 2019). We may find that the Tao method allows flexibility and moderation in choosing either of the two basic methods. Based on this comparison of all spectral methods, the pooling method is the most widely used of all differential equations (Awonusika, 2024; Mokhtary *et al*., 2016). It is used to solve higher order BVPs (Moghadam *et al*., 2022; Nagy, 2017). We applied a program based on the Fibonacci matrix to treat the nonlinear Klein-Gordon fractional equation. We used the fractional order of the first and second kind Legendre-integral-differential Fredholm equations and other forms of FDEs were dealt with using the explicit and implicit wave summation method in (Liu *et al*., 2022). We find that there are four types of Chebyshev polynomials in Jacobi polynomials. We find that all these types are represented in an organized manner, which is important for their use in various scientific applications. It plays an important role in numerical analysis and approximation theories. We used the first and second types to treat many types of differential equations. The authors in (Hassani *et al*., 2019) introduced some generalized Chebyshev polynomials. In addition, it has been used to deal with optimal control problems. A sophisticated type of multidimensional Chebyshev polynomials was presented in (Cesarano, 2019; Masjed-Jamei, 2007). In this paper, a type of orthogonal Chebyshev polynomial of the first kind is presented. Polynomials with variable coefficients are also introduced for this purpose. Polynomials work in practical and laboratory practice in programming, with a more accurate representation of the orthogonal relationship. We presented the proposed algorithm for processing (FDEs) and presented and verified numerical experiments to verify the effectiveness, efficiency and applicability of the proposed algorithm. Finally, the results are presented.

Some basic formulas of a series. There will be an overview of a series of classes of orthogonal polynomials. Some basic properties of fractional calculus are presented.

Definition of orthogonal polynomials through several aspects. In (Spanier, 1974), a polynomial solution to the differential equations:

$$(\alpha x^2+\beta x+\delta)\omega_i''(x)+(\gamma x+\rho)\omega_i'(x)-i\big((i-1)\alpha+\gamma\big)\omega_i(x)=0 \quad \text{.....................................(1)}$$

The study of polynomials Illustrate the recursive relationship between the second degree and the higher degrees, equation (1) and works to clarify and generalize some classical polynomials. It was clear in (Talaei and Asgheri, 2018) that the solution of first-order polynomials was given by the following formula:

$$\omega_i(x)=\omega_i^{\alpha,\beta,\delta,\rho}(x)=\sum_{i=0}^{n}\mathcal{X}_{k,i}(\alpha,\beta,\delta,\rho)x^k\,,\beta\,)\ldots \quad \text{..................(2)}$$

whereas the coefficients $(\alpha,\beta,\delta,\rho)$ are explicitly as follows:

$$\mathcal{X}_{k,i}(\alpha,\beta,\delta,\rho)=\binom{n}{k}\left(\frac{2\alpha}{\beta+\sqrt{\beta^2-4\alpha\delta}}\right)^{k-n}\times 2^{F_1}\left(k-n,\begin{matrix}\frac{2\alpha\delta-\beta\delta}{2\alpha\sqrt{\beta^2-4\alpha\delta}}+1-\frac{\delta}{2\alpha}-n\\ 2-\frac{\delta}{\alpha}-2n\end{matrix}\,\vdots\,\frac{2\sqrt{\alpha^2-4\alpha\rho}}{\left(\beta+\sqrt{\alpha^2-4\alpha\rho}\right)}\right)\ldots \quad \text{..................(3)}$$

we see that the coefficients (k) are binomial coefficients. In addition, 2^{F_1} which appears in equation (3) is a geometric function that represents a special case of the following:

$$\mathcal{A}^{F_r}\left(\begin{matrix}q_1,q_2,\cdots,q_n\,\vdots\\ z_1,z_2,\cdots,z_n\,\vdots\end{matrix}\quad x\right)=\sum_{n=0}^{\infty}\frac{((q_1)_n(q_2)_n\cdots(q_i)_n)x}{((z_1)_n(z_2)_n\cdots(z_i)_n)(n!)'} \quad \text{..................(4)}$$

where $\mathcal{A}$ and s are positive integers, there is no q_i is zero or a non-positive integer and the symbol z_i denotes the Pochhammer symbol. It was explained in (Chen *et al.*, 2012) that the general properties of the polynomials in equation (2) which is represented in exponent form in (2) were presented in (Bonab and Javidi, 2020). Such as the Rodriguez formula for $\omega_i^{\alpha,\beta,\delta,\rho}(x)$ given by

$$\omega_i^{\alpha,\beta,\delta,\rho}(x)=\frac{1}{\left(\frac{\delta}{\alpha}+n-1\right)\alpha^i\omega_i^{\alpha,\beta,\delta,\rho}(x)}\mathcal{D}^n(\alpha x^2+\beta x+\rho)^n\omega_i^{\alpha,\beta,\delta,\rho}(x) \quad \text{..................(5)}$$

$$\psi_i^{\alpha,\beta,\delta,\rho}(x)=e^{\left(\int\left(\frac{(\delta-2\alpha)x+(t-\beta)}{\alpha x^2+\beta x+\rho}\right)dx\right)}$$

We find in (Karunakar and Chaknaverty, 2019). that the polynomials, $\psi_i^{\alpha,\beta,\delta,\rho}(x)\geq 0$ orthogonal and the choices of the phenomena α, β, δ, ρ and e on the interval (Q, R) where Q and R are the zeros of the second-order equation: an $\alpha x^2 + \beta x + \rho = 0$. It is also clear that the following important formula:

$$\int_Q^R \omega_i^{\alpha,\beta,\delta,\rho}(x)\left(\psi_i^{\alpha,\beta,\delta,\rho}(x)\right)^2 dx = \frac{(-1)^n n!}{\left(\frac{\delta}{\alpha}+n-1\right)_n \alpha^n}\int_Q^R (x^2+\beta\, x+\rho)^n\, e^{\left(\int\left(\frac{(\delta-2\alpha)x+(t-\beta)}{\alpha x^2+\beta\, x+\rho}\right)dx\right)}dx \quad \text{.......(6)}$$

Remark 2-1. We point out here that there are many phenomena in polynomials $\psi_i^{\alpha,\beta,\delta,\rho}(x)$. The existence of a few orthogonal polynomial series, including Jacobi and Laguerre polynomials, means that they are special series of polynomials (Karunakar and Chaknaverty, 2019). This shows the importance of this study.

Definition 2-1. The Riemann-Liouville fractional integral operator $\mathcal{W}_x^u$ of order u>0 can be defined as follows:

$$\mathcal{W}_x^u \mathcal{X}(x) = \frac{1}{\Gamma(u)}\int_0^x \left(\frac{\mathcal{X}(\lceil u\rceil)(\ell)}{\left((x-\ell)^{u+1-\lceil u\rceil}\right)}\right) d\ell \quad \text{.......(7)}$$

Definition 2-2. The Caputo fractional derivative operator $\mathcal{W}_x^u \mathcal{X}(x)$ of the order $u > 0$ can be defined as follows:

$$\mathcal{D}_x^u \mathcal{X}(x) = \frac{1}{\Gamma(\lceil u\rceil - u)}\int_0^x \frac{\mathcal{X}^{(\lceil u\rceil)}(t)}{(x-t)^{u+1-\lceil u\rceil}}\, dt, \quad \text{.......(8)}$$

where $\lceil u\rceil$ Indicates a function and x>0.

Chebyshev polynomials of the first kind. We have introduced a set of polynomials that will be generalized to sets of Chebyshev polynomials of the first kind. We also introduce some basic properties of Chebyshev polynomials, and we will also introduce generalized Chebyshev polynomials, and some basic properties will be developed in them.

Generalized Chebyshev polynomials of the first kind. We will extract the generalized group from Chebyshev polynomials. It is a special set of sets of polynomials $\omega_i^{\alpha,\beta,\delta,\rho,\ell}(x)$ in equation (2). We will make the following choices:

$$\alpha = -1, \rho = \beta + 1, \delta = -1, \ell = 0 \quad \text{.......(9)}$$

we have the free parameter b. We will denote the resulting polynomials $H_n^\beta(x)$ by:

$$H_n^\beta(x) = \sum_{i=0}^n \mathcal{X}_{k,i}(-1,\beta,\beta+1,-1,0)\,\mathcal{X}^n$$

we find that the polynomials $H_n^\beta(x)$ are orthogonal to $[-1, \beta+1]$ with respect to the following weight function $W(x)$:

$$W(x) = (\beta - x + 1)^{-\frac{1}{\beta+2}}(1+x)^{-\left(\frac{\beta+1}{\beta+2}\right)} \quad \text{.......(10)}$$

From the formula in equation (5) for the polynomials $\omega_i^{\alpha,\beta,\delta,\rho}(x)$, it can be shown that the formula $H_n^\beta(x)$ is given by:

$$H_n^\beta(x) = \frac{2(-1)^n n!}{(2!)!\; W(x)}\mathcal{D}^n\left((\beta - x + 1)^n (1+x)^n\, W(x)\right), n \geq 1 \quad \text{.......(11)}$$

We find here that we have two main reasons for choosing the five phenomena as in equation (9):

Lemma 3-1. For every positive integer k and every positive real number β, the integral formula holds:

$$\int_{-1}^{\beta+1}(1+\beta+\beta x - x^2)^k\, W(x)dx = \frac{(\beta+2)^{2k}\,\Gamma\left(\left(\frac{1}{\beta+2}\right)+k\right)\Gamma\left(\left(\frac{\beta+1}{\beta+2}\right)+k\right)}{\left((2k)!\right)} \quad \text{.......(12)}$$

Proof. To simplify the integration, we replace x with $(\beta+2)^{2k}$, that is, we have the following formula.

$$\int_{-1}^{\beta+1}(1+\beta+\beta x - x^2)^k\, W(x)dx = (\beta+2)^{2k}\int_0^1 \left((1-x)^{-\left(\left(\frac{1}{\beta+2}\right)-k\right)} x^{-1+\left(\frac{1}{\beta+2}\right)+k}\right) dx \quad \text{......(13)}$$

It is easy to prove that the following equation holds:

$$\int_0^1 (1-x)^{k-\left(\frac{1}{\beta+2}\right)}\, x^{k-\left(\frac{\beta+1}{\beta+2}\right)} dx = \frac{\Gamma\left(\left(\frac{1}{\beta+2}\right)+k\right)\Gamma\left(\left(\frac{\beta+1}{\beta+2}\right)+k\right)}{\left((2k)!\right)} \quad \text{.......(14)}$$

From this, we arrive at the following identity.

$$\int_{-1}^{\beta+1}(1+\beta+\beta x - x^2)^k\, W(x)dx = \frac{(\beta+2)^{2k}\,\Gamma\left(\left(\frac{1}{\beta+2}\right)+k\right)\Gamma\left(\left(\frac{\beta+1}{\beta+2}\right)+k\right)}{\left((2k)!\right)} \quad \text{.......(15)}$$

Theorem 3-1. For every positive integer k and every positive real number β then the integrated formula holds:

$$\int_{-1}^{\beta+1}\left(H_n^{\beta}(x)\right)^2 W(x)dx = \frac{2^{1-4k}(\beta+2)^{2k}\pi\,\Gamma\left(\left(\frac{1}{\beta+2}\right)+k\right)\Gamma\left(\left(\frac{\beta+1}{\beta+2}\right)+k\right)}{\Gamma\left(k+\frac{1}{2}\right)^2} \qquad (16)$$

where the weight function w(x) is given by equation (10).

Proof. The proof of equation (12) depends on making use of formula (6). More precisely and clearly, if we substitute α = –1, ρ = β, δ = –1, ℓ = 0, then we get:

$$\int_{-1}^{\beta+1}\left(H_n^{\beta}(x)\right)^2 W(x)dx = \frac{2(k!)^2}{(2k)!}\int_{-1}^{\beta+1}\left((1+\beta+\beta x-x^2)^k\, W(x)\right)dx \qquad (17)$$

Directly applying Lemma 1 produces formula (16).

Shifted generalized chebyshev polynomials. In many practical applications, we can identify. Aligned polynomials on the interval $[0,1]$. Then we define the transformed polynomial $RH_n^{\beta}(x)$ on $[0,1]$ as:

$$RH_n^{\beta}(x) = H_n^{\beta}(\beta+2)x-1 \qquad (18)$$

The Rodriguez formula for Chebyshev polynomials $H_n^{\beta}(x)$ in (11). It can simply be transformed to give the counterpart of the Chebyshev polynomials defined in (18) (Karunakar and Chaknaverty, 2019; Aghigh *et al*., 2008). The polynomial $RH_n^{\beta}(x)$ can be constructed using Rodriguez's formula:

$$RH_n^{\beta}(x) = \frac{2(-1)^k k!\,(\beta+2)^k}{(2k)!\,\widetilde{W}(x)}\mathcal{D}^k\left((1-x)^k x^k \widetilde{W}(x)\right) \qquad (19)$$

where $k \geq 1$ and $\widetilde{W}(x)$ is given by:

$$\widetilde{W}(x) = \frac{1}{\beta+2}(x)^{\left(\frac{1}{\beta+2}-1\right)}(1-x)^{-\left(\frac{1}{\beta+2}\right)} \qquad (20)$$

for further studies, it is useful to define some basic properties of the transformed Chebyshev polynomials $RH_n^{\beta}(x)$.

Theorem 3-2. For every integer mu r, it has the following formula:

$$x^q = \sum_{i=0}^{q}\left(\frac{(\beta+2)^{k-q}(1+k-i)}{i!\,(2q-2i+1)_i}\right)\left(RH_{q-j}^{\beta}(x)\right) \qquad (21)$$

Proof. First, we can write the identity:

$$x^n = \sum_{i=0}^{n}\left(\mathcal{X}_{k,i} RH_{n-j}^{\beta}\right)(x) \qquad (22)$$

To prove the identity of (21), we need to find the coefficients $\mathcal{X}_{k,i}$ Now, multiplying both sides of (22) by $RH_{n-j}^{\beta}(x)\widetilde{W}(x)$ and integrating from 0 to 1, we get:

$$\sum_{i=0}^{n}\mathcal{X}_{k,i}\int_0^1 RH_{n-j}^{\beta}(x)\, RH_m^{\beta}(x)\widetilde{W}(x)dx = \int_0^1\left(x^n RH_m^{\beta}(x)\widetilde{W}(x)\right)dx \qquad (23)$$

The orthogonality of $RH_{n-j}^{\beta}(x)$ in (20) enables us to determine the coefficients $\alpha,\ i,\ n$ in the model:

$$\mathcal{X}_{k,i} = \frac{1}{\psi_{n-i}\Psi_{n-i}}\int_0^1\left(x^n RH_{n-j}^{\beta}(x)\widetilde{W}(x)\right)\, dx \qquad (24)$$

where the variable q is as defined in (21). Now, from the force representation of $RH_{n-j}^{\beta}(x)$ in (18), we can write equations:

$$RH_n^{\beta}(x) = \sum_{m=0}^{k} H_{m,k}\, x^{k-m} \qquad (25)$$

where we give the coefficients RH_{n-j}^{β} by (18). Substituting formula (25) into formula (24), we can write the coefficients $\mathcal{X}_{k,i}$ in form:

$$\mathcal{X}_{k,i} = \frac{1}{\psi_{n-i}\Psi_{n-i}}\sum_{m=0}^{k-1} H_{m,k-i}\int_0^1\left(x^{2k-i-m}\widetilde{W}(x)\right)dx. \qquad (26)$$

It is possible to obtain the following equation:

$$\int_0^1 x^k\, \widetilde{W}(x)dx = \frac{\Gamma\left(\frac{\beta+1}{\beta+2}\right)\Gamma\left(\frac{1}{\beta+2}+k\right)}{(\beta+2)k!} \qquad (27)$$

Applying theorem 3-1 puts the coefficients $\mathcal{X}_{k,i}$ in the following form:

$$\mathcal{X}_{k,i} = \frac{(\beta+2)^{i-k}(1+k-i)_i\left(\left(\frac{1}{\beta+2}+k-i\right)_i\right)}{i!\,(1+2k-2i)_i} \text{........................(28)}$$

This proves theorem 2.

Remark 3.1. Where we find non-homogeneous linear polynomial FDEs (26) with initial conditions, identity:

$$\mathcal{X}^m(0) = \mathcal{Y}_m, \qquad m = 0,1,\cdots,k-1$$

where:
$\mathcal{Y}_m$ are arbitrary constants, $0 \le m \le n-1$, the following transformation is used:

$$\tilde{\mathcal{X}}(x) = \mathcal{X}(x) - \sum_{m=0}^{n-1} \frac{\mathcal{Y}_m}{m!}\mathcal{X}^m \text{.............................(29)}$$

Illustrative polynomials and comparisons. Here we limit ourselves to testing our proposed algorithm. For this reason, we will present two numerical examples and compare them with other techniques to demonstrate the accuracy and high efficiency of the proposed algorithm.

Example 5.1. Consider the partial oscillation equation of a vehicle immersed in a Newtonian fluid with the condition (Bonab and Javidi, 2020; Chen *et al.*, 2012):

$$\mathcal{D}_\alpha^u\mathcal{X}(\varphi) + \mathcal{X}(\varphi) = \varphi^4 - \frac{1}{2}\varphi^3 - \frac{3}{\Gamma(4-u_1)}\varphi^{3-u_1} + \frac{24}{\Gamma(5-u_1)}\varphi^{4-u_1} \qquad 0 < u_1 < 1,\ \mathcal{X}(0) = 0.$$

The solution to the problem is:

$$\mathcal{X}(\varphi) = \varphi^4 - \frac{1}{2}(\varphi)^3$$

The solution of this problem numerically in (Chen *et al.*, 2012) is based on the Chelyshkov Spectral Method for the numerical solution of the problem, while the method was applied in (Bonab and Javidi, 2020), for the numerical treatment of the problem. The error of the presented method for different values of N with μ_1=0.25 and b=2 is reviewed in Table 1. Moreover, the results are compared in Table 2 with those obtained by (Chen *et al.*, 2012; Bonab and Javidi, 2020) and we get the results that are in the table to ensure the verification of the proposed method compared to the other method. Figure 1 plots the maximum absolute error of the solutions generated by applying our proposed algorithm for μ_1=0.25, β=0 and N=4, while Fig. 2 displays the proposed algorithm for the case corresponding to μ_1=0.75 and β=2 with distinct values of N.

Remark 5.1. From the data in Table 2, we conclude that standard Chebyshev polynomials of the first kind

Table 1. Comparison of $\|.\|_\infty$ −errors where ℓ_∞ are given max $|.|$ or max errors and $\|.\|_2$-errors, ℓ_2 are given $\sqrt{\sum_{i=1}^{n}(.)_i^2}$, of our algorithm at μ_1=0.75 and N=4 with distinct β for example 5.1

β	$\|.\|_\infty$-Errors	$\|.\|_2$-Errors
0	3.02364×10^{-5}	1.11641×10^{-5}
1	4.12031×10^{-5}	2.57854×10^{-5}
2	5.99751×10^{-5}	3.67207×10^{-6}
3	1.64375×10^{-6}	5.79251×10^{-6}
4	1.64266×10^{-5}	5.65481×10^{-6}
5	1.64236×10^{-5}	5.65345×10^{-6}

Table 2. Comparison of $\|.\|_2$-errors of our algorithm at μ_1=0.75 and β= for distinct N with the Chelyshkov Spectral Method (Chen *et al.*, 2012) and proposed algorithm (Bonab and Javidi, 2020) for example 5.1

N	CCSM	Our method
8	3.02×10^{-3}	8.50×10^{-2}
16	3.12×10^{-2}	7.60×10^{-2}
32	3.82×10^{-1}	5.21×10^{-2}

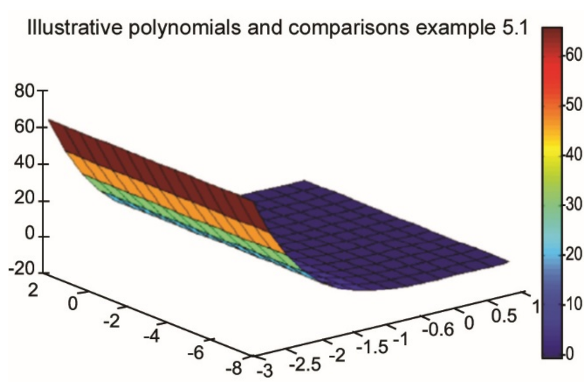


Fig. 1. $\mathcal{X}_N(\varphi)$ of our algorithm for $u_1 = 0.25$ and $N = 4$ with $\beta = 0$ for example 5.1.

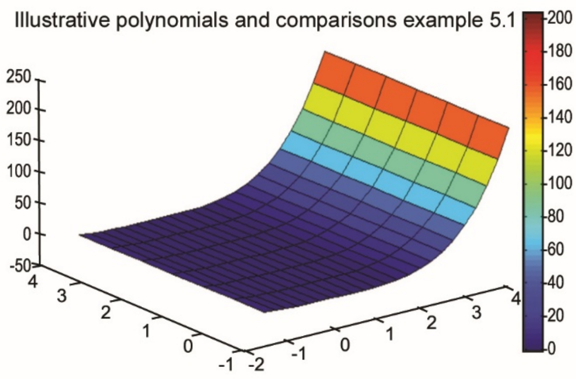


Fig. 2. $(\mathrm{Log10})\|.\|_{\infty}-$ errors) of our algorithm at μ_1=0.25 and β=2 with distinct N for example 5.1.

are not the best approximations among the different classes of shifted polynomials $RH_n^{\beta}(x)$. This demonstrates the importance of generalizing to the first type of Chebyshev polynomials and the displacement operations on them, as well as the effect of the parameter β that occurs in shifted polynomials (Bonab and Javidi, 2020; Chen *et al.*, 2012).

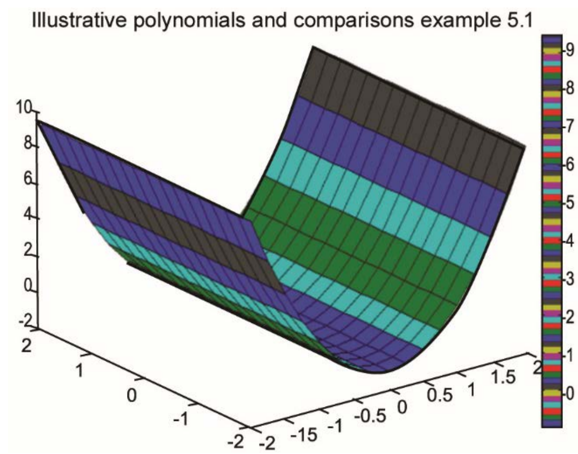
Example 5.2. Consider the following problem FDE:

$$\mathcal{D}_{\alpha}^{u}\mathcal{X}(\varphi)+\mathcal{X}(\varphi)=\varphi^2+\frac{2}{\Gamma(3-u_1)}\left(\varphi^{(2-u_1)}\right) \text{ and}$$
$$\mathcal{X}(0)=0,$$

in which $\mathcal{X}(\varphi)=\varphi^2$ is the exact solution. Methods have been developed to solve this problem numerically. Which was proposed in (Talaei and Asgari, 2018; Chen *et al.*, 2012). There are explicit methods based on the third-order fractional inverse differentiation method for numerical solutions. We applied a special algorithm to obtain the numerical solution to this problem. In Table 3, the errors$\|.\|_{\infty}$ resulting from the application of the proposed algorithm are shown corresponding to μ_1=0.25 and N=2 with distinct b. In addition to, in third ordered the comparison between the $\|.\|_{\infty}$-Errors resulting from the proposed algorithm for μ_1=0.6 and β=1 for N=2 with the explicit methods based on the third-order developed in (Talaei and Asgari, 2018; Chen *et al.*, 2012), (h is the mesh size), we explain the effect of the parameter β and the comparison in Table 4 is made. In Fig. 3 the resulting $((\mathrm{Log10})\|.\|_{\infty}-$ errors) for the proposed algorithm for β=0 and β=1, N=2 with distinct μ_1. Fig. 4 give the $\mathcal{X}_N(\varphi)$ for the algorithm respectively: μ_1=0.6, β=1, N=2 and μ_1=0.7, β=1, N=2.

Table 3. $\|.\|_{\infty}-$ Errors of our algorithm at μ_1=0.6 and N=2 with distinct β for example 5.2.

β	$\|.\|_{\infty}$-Errors
0	2.00310×10^{-11}
1	1.1016×10^{-11}
2	2.0320×10^{-11}
3	2.0310×10^{-11}
4	2.03210×10^{-11}
5	2.03210×10^{-11}



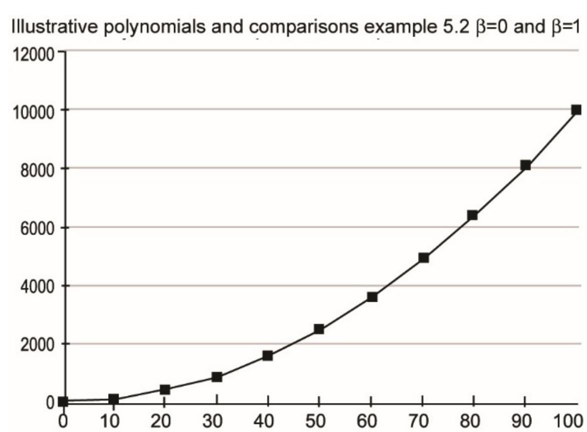


Fig. 3. Comparison of $((\mathrm{Log10})\|.\|_{\infty}-$ Errors) of our algorithm at β=0 and β=1 for N=0 with distinct u1 for example 2.

Table 4. Comparison of $(\|.\|_\infty - Errors)$of our algorithm at $u_1 = 0.7, 0.8$, and $\beta = 1$ for $N = 2$ with the for Example 5.2.

FBDM			Our Method	
u_1	h=0.1	h=0.01	h=0.001	N=2
0.6	0.50×10^{-2}	0.17×10^{-2}	0.32×10^{-3}	6.62×10^{-5}
0.8	0.30×10^{-2}	0.32×10^{-2}	0.52×10^{-3}	3.12×10^{-5}

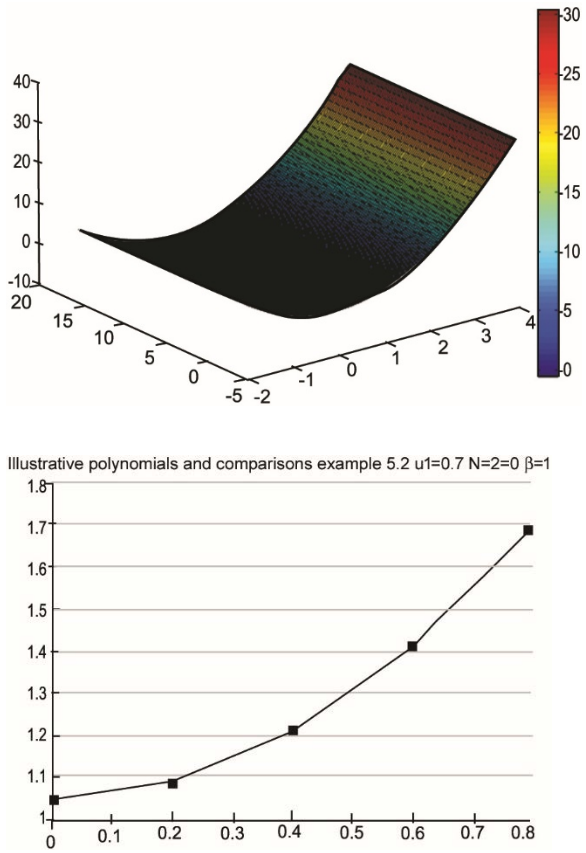


Fig. 4. MAE of $\mathcal{X}_N(\varphi)$ our algorithm at μ_1=0.7 and β=1 with N=2 for example 5-2.

Conclusions

The spectral method was used to treat polynomial FDEs with initial conditions and transformed Chebyshev polynomials of the first type that was used primarily based on a linear matrix system and used the appropriate software solution, in addition, some test examples are presented to verify the effectiveness of the algorithms accuracy. Chebyshev's polynomials are not always better than other polynomials. Generalized polynomials are introduced, differential equations are solved using generalized polynomials. We intend to use orthogonal polynomials to solve other types of differential equations.

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